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NONAXISYMMETRIC ELASTOPLASTIC IMPACT OF A PARABOLIC BODY ON A SPHERICAL SHELL

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A method is proposed to calculate a spherical shell under nonaxisymmetric impact of a massive body. The motion of the shell is described by momentless equations, which are solved using the Laplace transformation and an asymptotic expansion of the required quantities in a small parameter. The contact interaction force $P(t)$ was determined for the elastoplastic model of local bearing deformation for a parabolic impactor. Plots of the solution are given. The validity of the results is confirmed by good agreement between the solution and the limiting cases — an axisymmetric impact and an impact on a half-space.

Key words: *contact problem, elastoplastic impact, parabolic body, spherical shell.*

Let us consider a nonaxisymmetric normal impact of a massive body on a circular sector of a contour-hinged spherical shell. The general displacements of the shell are considered elastic, and its local displacements in the zone of contact of the body with the shell are considered elastoplastic. At the initial time, the shell is at rest and the body has a velocity V_0 that is much lower than the velocity of elastic waves in the shell. This allows us to ignore the inertia of local bearing deformation in the contact region. As a result, the dependence of the local bearing strain α on the contact force P can be determined in a similar manner as in the static problem.

We use coordinate axes directed along a meridian φ and a parallel θ. The impact is performed at a point $(\varphi_1, 0)$ by a body of mass m with elastic constants E_2 and ν_2, a plastic constant k_2, and a curvature radius at the contact point R_2. The apex angle of the shell arc is φ_0.

Denoting the normal displacement of the shell at the contact point by w and the displacement of the impacting body by s, we have the following relation [1]:

$$s = w + \alpha. \tag{1}$$

The displacement of the impactor s is determined by integrating the differential equation of motion for the body $m\ddot{s} = -P(t)$ subject to the initial conditions $s_0 = 0$ and $\dot{s}_0 = V_0$:

$$s(t) = V_0 t - \frac{1}{m} \int_0^t \int_0^{t_1} P(t_2)\, dt_1\, dt_2. \tag{2}$$

The displacement of the shell under the action of the force $P(t)$ is determined from the momentless equations of motion for spherical shells [2]:

$$\begin{gathered}
(N_\varphi \sin\varphi)_{,\varphi} + N_{\varphi\theta,\theta} - N_\theta \cos\varphi = \rho h R_1 \ddot{u}_\varphi \sin\varphi,\\
(N_{\varphi\theta} \sin\varphi)_{,\varphi} + N_{\theta,\theta} + N_{\varphi\theta} \cos\varphi = \rho h R_1 \ddot{u}_\theta \sin\varphi,\\
N_\varphi + N_\theta = -\rho h R_1 \ddot{w} + q_3 R_1.
\end{gathered} \tag{3}$$

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Here

$$N_\varphi = E_1h((1-\nu_1^2)R_1)^{-1}(u_{\varphi,\varphi} + w + \nu_1(u_\varphi \cot\varphi + u_{\theta,\theta}\sin^{-1}\varphi + w)),$$

$$N_\theta = E_1h((1-\nu_1^2)R_1)^{-1}(u_\varphi \cot\varphi + u_{\theta,\theta}\sin^{-1}\varphi + w + \nu_1(u_{\varphi,\varphi} + w)),$$

$$N_{\varphi\theta} = E_1h(2(1+\nu_1)R_1)^{-1}(u_{\theta,\varphi} - u_\theta \cot\varphi + u_{\varphi,\theta}\sin^{-1}\varphi)$$

(ρ is the density of the material, h and R_1 are the thickness and radius of the shell, q_3 is the load, and E_1 and ν_1 are the elastic constants of the shell). The plastic constant of the shell is denoted by k_1.

The boundary conditions are given by

$$u_\varphi\Big|_{\varphi=\varphi_0} = 0, \qquad w\Big|_{\varphi=\varphi_0} = 0. \tag{4}$$

We introduce the following dimensionless quantities:

$$v = \frac{u_\varphi}{R_1}, \qquad u = \frac{u_\theta}{R_1}, \qquad w = \frac{w}{R_1}, \qquad \tau = \frac{tc}{R_1}, \qquad c^2 = \frac{E_1}{(1-\nu_1^2)\rho}.$$

Then, after elimination of the forces N_φ, N_θ, and $N_{\varphi\theta}$, system (3) becomes

$$v_{,\varphi\varphi}\sin\varphi + 0.5(1-\nu_1)v_{,\theta\theta}\sin^{-1}\varphi + 0.5(1+\nu_1)u_{,\theta\varphi} + v_{,\varphi}\cos\varphi$$

$$-(\cot\varphi\cos\varphi + \nu_1\sin\varphi)v - 0.5(3-\nu_1)u_{,\theta}\cot\varphi + (1+\nu_1)w_{,\varphi}\sin\varphi = v_{,\tau\tau}\sin\varphi,$$

$$0.5(1-\nu_1)u_{,\varphi\varphi}\sin\varphi + u_{,\theta\theta}\sin^{-1}\varphi + 0.5(1+\nu_1)v_{,\theta\varphi} + 0.5(1-\nu_1)u_{,\varphi}\cos\varphi$$

$$+0.5(1-\nu_1)(\sin\varphi - \cot\varphi\cos\varphi)u + 0.5(3-\nu_1)v_{,\theta}\cot\varphi + (1+\nu_1)w_{,\theta} = u_{,\tau\tau}\sin\varphi,$$

$$(1+\nu_1)(v_{,\varphi} + v\cot\varphi + 2w + u_{,\theta}\sin^{-1}\varphi) = -w_{,\tau\tau} + q,$$

where $q = (1-\nu_1^2)(E_1h)^{-1}R_1q_3$.

Let us make the replacements $V = v\sin\varphi$ and $U = u\sin\varphi$:

$$V_{,\varphi\varphi} + 0.5(1-\nu_1)V_{,\theta\theta}\sin^{-2}\varphi + 0.5(1+\nu_1)U_{,\theta\varphi}\sin^{-1}\varphi - V_{,\varphi}\cot\varphi$$

$$+(1-\nu_1)V - 2U_{,\theta}\cos\varphi\sin^{-2}\varphi + (1+\nu_1)w_{,\varphi}\sin\varphi = V_{,\tau\tau},$$

$$0.5(1-\nu_1)U_{,\varphi\varphi} + U_{,\theta\theta}\sin^{-2}\varphi + 0.5(1+\nu_1)V_{,\theta\varphi}\sin^{-1}\varphi - 0.5(1-\nu_1)U_{,\varphi}\cot\varphi$$

$$+(1-\nu_1)U + (1-\nu_1)V_{,\theta}\cos\varphi\sin^{-2}\varphi + (1+\nu_1)w_{,\theta} = U_{,\tau\tau},$$

$$(1+\nu_1)(V_{,\varphi}\sin^{-1}\varphi + 2w + U_{,\theta}\sin^{-2}\varphi) = -w_{,\tau\tau} + q.$$

Applying the Laplace transformation over time t and denoting the transforms of V, U, w, and q by V^*, U^*, w^*, and q^*, respectively, we obtain

$$V^*_{,\varphi\varphi} + 0.5(1-\nu_1)V^*_{,\theta\theta}\sin^{-2}\varphi + 0.5(1+\nu_1)U^*_{,\theta\varphi}\sin^{-1}\varphi$$

$$-V^*_{,\varphi}\cot\varphi + (1-\nu_1-p^2)V^* - 2U^*_{,\theta}\cos\varphi\sin^{-2}\varphi + (1+\nu_1)w^*_{,\varphi}\sin\varphi = 0,$$

$$0.5(1-\nu_1)U^*_{,\varphi\varphi} + U^*_{,\theta\theta}\sin^{-2}\varphi + 0.5(1+\nu_1)V^*_{,\theta\varphi}\sin^{-1}\varphi - 0.5(1-\nu_1)U^*_{,\varphi}\cot\varphi \tag{5}$$

$$+(1-\nu_1-p^2)U^* + (1-\nu_1)V^*_{,\theta}\cos\varphi\sin^{-2}\varphi + (1+\nu_1)w^*_{,\theta} = 0,$$

$$(1+\nu_1)(V^*_{,\varphi}\sin^{-1}\varphi + (2+p^2(1+\nu_1)^{-1})w^* + U^*_{,\theta}\sin^{-2}\varphi) = q^*.$$

From the third equation, we obtain the following expression for $U^*_{,\theta}$:

$$U^*_{,\theta} = (q^* - (1+\nu_1)V^*_{,\varphi}\sin^{-1}\varphi - (2(1+\nu_1)+p^2)w^*)(1+\nu_1)^{-1}\sin^{-2}\varphi.$$

We substitute $U^*_{,\theta}$ into (5), previously differentiating the second equation with respect to θ. As a result, the system becomes

$$0.5(1-\nu_1)V^*_{,\varphi\varphi} + 0.5(1-\nu_1)V^*_{,\theta\theta}\sin^{-2}\varphi + 0.5(1-\nu_1)V^*_{,\varphi}\cot\varphi + (1-\nu_1-p^2)V^*$$

$$-0.5p^2 w^*_{,\varphi}\sin\varphi + (1-\nu_1)(2+p^2(1+\nu_1)^{-1})w^*\cos\varphi + 0.5q^*_{,\varphi}\sin\varphi - (1-\nu_1)(1+\nu_1)^{-1}q^*\cos\varphi = 0,$$

$$-0.5(1-\nu_1^2)V^*_{,\varphi\varphi\varphi}\sin\varphi - 0.5(1-\nu_1^2)V^*_{,\theta\theta\varphi}\sin^{-1}\varphi + (1-\nu_1^2)V^*_{,\theta\theta}\cot\varphi \tag{6}$$

$$-0.5(1-\nu_1^2)V^*_{,\varphi\varphi}\cos\varphi + ((1+\nu_1)p^2\sin\varphi + 0.5(1-\nu_1^2)(\cos^2\varphi - \sin^2\varphi)\sin^{-1}\varphi)V^*_{,\varphi}$$

$$-0.5(1-\nu_1)(2(1+\nu_1)+p^2)w^*_{,\varphi\varphi}\sin^2\varphi + ((1+\nu_1)^2 - (2(1+\nu_1)+p^2))w^*_{,\theta\theta}$$

$$-1.5(1-\nu_1)(2(1+\nu_1)+p^2)w^*_{,\varphi}\sin\varphi\cos\varphi + p^2(2(1+\nu_1)+p^2)w^*\sin^2\varphi$$

$$+0.5(1-\nu_1)q^*_{,\varphi\varphi}\sin^2\varphi + q^*_{,\theta\theta} + 1.5(1-\nu_1)q^*_{,\varphi}\sin\varphi\cos\varphi - p^2q^*\sin^2\varphi = 0.$$

The solution of (6) is sought as series in Legendre polynomials that satisfy boundary conditions (4):

$$w^* = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} w_{2n+1m}P_{2n+1}(\cos\delta_1\varphi)\cos m\theta,$$

$$V^* = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} V_{2n+1m}P_{2n+1}(\cos\delta_1\varphi)\cos m\theta, \qquad \delta_1 = \pi/(2\varphi_0).$$

The load $q(t,\varphi,\theta)$ from the concentrated force $P(t)\delta(\varphi-\varphi_1)\delta(\theta-0)$ is also expanded in a series in Legendre polynomials:

$$q = \frac{P(t)}{2\pi R_1^2(1-\cos\varphi_0)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(4n+3)P_{2n+1}(\cos\delta_1\varphi_1)P_{2n+1}(\cos\delta_1\varphi)\cos m\theta,$$

$$q^* = \frac{P^*(p)}{2\pi R_1^2(1-\cos\varphi_0)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(4n+3)P_{2n+1}(\cos\delta_1\varphi_1)P_{2n+1}(\cos\delta_1\varphi)\cos m\theta.$$

Substituting the expansions of w^*, V^*, and q^* into (6) and using orthogonality property of the system of cosines on the segment $[-\pi,\pi]$, we obtain

$$0.5(1-\nu_1)\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1,\varphi\varphi} - 0.5(1-\nu_1)m^2\sin^{-2}\varphi\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1}$$

$$+0.5(1-\nu_1)\cot\varphi\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1,\varphi} + (1-\nu_1-p^2)\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1}$$

$$-0.5p^2\sin\varphi\sum_{n=0}^{\infty}w_{2n+1m}P_{2n+1,\varphi} + (1-\nu_1)(2+p^2(1+\nu_1)^{-1})\cos\varphi\sum_{n=0}^{\infty}w_{2n+1m}P_{2n+1}$$

$$+0.5C\sin\varphi\sum_{n=0}^{\infty}(4n+3)P_{2n+1}(\cos\delta_1\varphi_1)P_{2n+1,\varphi}$$

$$-(1-\nu_1)(1+\nu_1)^{-1}C\cos\varphi\sum_{n=0}^{\infty}(4n+3)P_{2n+1}(\cos\delta_1\varphi_1)P_{2n+1} = 0,$$

$$-0.5(1-\nu_1^2)\sin\varphi\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1,\varphi\varphi\varphi} + 0.5(1-\nu_1^2)m^2\sin^{-1}\varphi\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1,\varphi}$$

$$-(1-\nu_1^2)m^2\cot\varphi\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1} - 0.5(1-\nu_1^2)\cos\varphi\sum_{n=0}^{\infty}V_{2n+1m}P_{2n+1,\varphi\varphi}$$

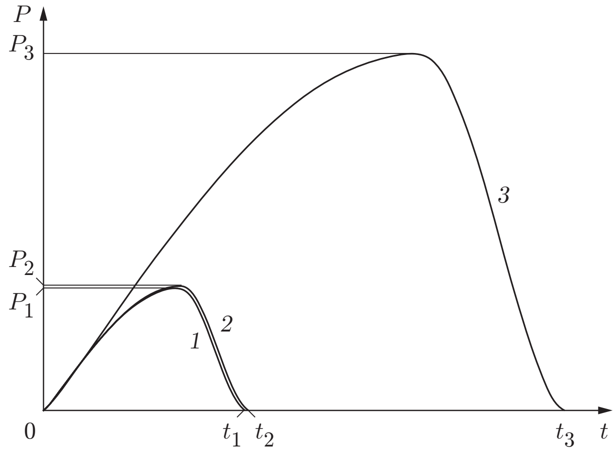

Fig. 1. Functions $P(t)$ for $\varphi_1 = 0.001$ rad: $P_1 = 18.5$ kN, $P_2 = 18.9$ kN, $P_3 = 54.4$ kN, $t_1 = 4.165 \cdot 10^{-5}$ sec, $t_2 = 4.266 \cdot 10^{-5}$ sec, and $t_3 = 10.767 \cdot 10^{-4}$ sec.

$$
\begin{aligned}
&+ ((1+\nu_1)p^2 \sin\varphi + 0.5(1-\nu_1^2)(\cos^2\varphi - \sin^2\varphi)\sin^{-1}\varphi) \sum_{n=0}^{\infty} V_{2n+1m} P_{2n+1,\varphi} \\
&- 0.5(1-\nu_1)(2(1+\nu_1)+p^2)\sin^2\varphi \sum_{n=0}^{\infty} w_{2n+1m} P_{2n+1,\varphi\varphi} + ((1+\nu_1)^2 - (2(1+\nu_1)+p^2))m^2 \sum_{n=0}^{\infty} w_{2n+1m} P_{2n+1} \\
&- 1.5(1-\nu_1)(2(1+\nu_1)+p^2)\sin\varphi\cos\varphi \sum_{n=0}^{\infty} w_{2n+1m} P_{2n+1,\varphi} + p^2 \sin^2\varphi (2(1+\nu_1)+p^2) \sum_{n=0}^{\infty} w_{2n+1m} P_{2n+1} \\
&+ 0.5(1-\nu_1) C \sin^2\varphi \sum_{n=0}^{\infty} (4n+3) P_{2n+1}(\cos\delta_1\varphi_1) P_{2n+1,\varphi\varphi} - m^2 C \sum_{n=0}^{\infty} (4n+3) P_{2n+1}(\cos\delta_1\varphi_1) P_{2n+1} \\
&+1.5(1-\nu_1) C \sin\varphi\cos\varphi \sum_{n=0}^{\infty} (4n+3) P_{2n+1}(\cos\delta_1\varphi_1) P_{2n+1,\varphi} \\
&-p^2 C \sin^2\varphi \sum_{n=0}^{\infty} (4n+3) P_{2n+1}(\cos\delta_1\varphi_1) P_{2n+1} = 0.
\end{aligned} \tag{7}
$$

Here $C = P^*(p)(2\pi R_1^2(1-\cos\varphi_0))^{-1}$ and $P_{2n+1} = P_{2n+1}(\cos\delta_1\varphi)$.

The coefficients V_{2n+1m} and w_{2n+1m} are obtained using the method of a small parameter $\varepsilon = p^{-2}$. We note that using this method, Kil'chevskii studied impact on an arbitrary infinite shell without boundary conditions and obtained a number of qualitative results [3]. We represent the required quantities as

$$V_{2n+1m}(p) = V_{2n+1m}^0 \varepsilon^0 + V_{2n+1m}^1 \varepsilon^1 + V_{2n+1m}^2 \varepsilon^2 + O(\varepsilon^3),$$

$$w_{2n+1m}(p) = w_{2n+1m}^0 \varepsilon^0 + w_{2n+1m}^1 \varepsilon^1 + w_{2n+1m}^2 \varepsilon^2 + O(\varepsilon^3).$$

Substituting these expansions into system (7) and collecting the coefficients at identical powers of ε, we obtain V_{2n+1m}^i and w_{2n+1m}^i $(i = 0, 1, 2)$. For w_{2n+1m}, we have

$$w_{2n+1m}(p) = C(4n+3)P_{2n+1}(\cos\delta_1\varphi_1)(p^{-2} - 2(1+\nu_1)p^{-4}) + O(\varepsilon^3).$$

Using the inverse Laplace transformation for the displacement of the shell w and taking into account the first three terms of the expansion in $(\tau - \tau_1)$, we have

$$
\begin{aligned}
w(\varphi, \theta, \tau) = \frac{1-\nu_1^2}{2\pi h E_1 R_1 (1-\cos\varphi_0)} \int_0^{\tau} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (4n+3) P(\tau_1) \Big((\tau - \tau_1) - \frac{2}{3!}\,(1+\nu_1)(\tau-\tau_1)^3 \Big) \\
\times P_{2n+1}(\cos\delta_1\varphi_1) P_{2n+1}(\cos\delta_1\varphi) \cos m\theta \, d\tau_1.
\end{aligned} \tag{8}
$$

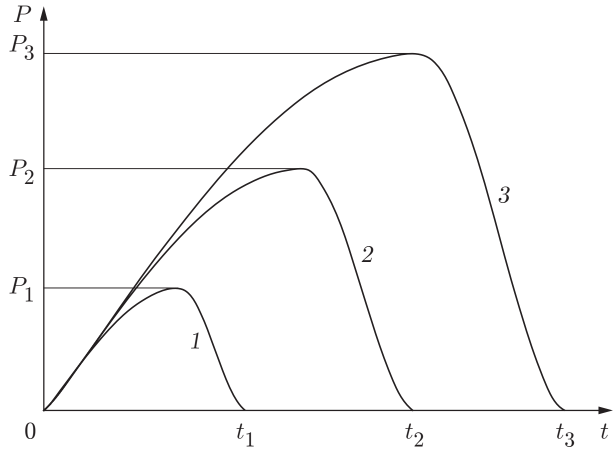

Fig. 2

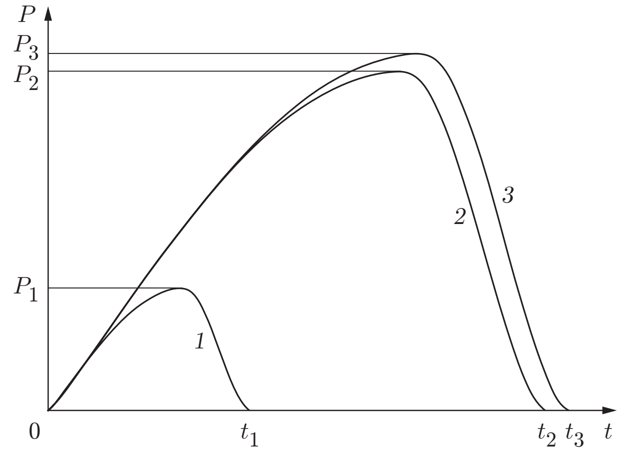

Fig. 3

Fig. 2. Functions $P(t)$ for $\varphi_1 = 0.01$ rad: $P_1 = 18.5$ kN, $P_2 = 36.7$ kN, $P_3 = 54.4$ kN, $t_1 = 4.165 \cdot 10^{-5}$ sec, $t_2 = 7.720 \cdot 10^{-5}$ sec, and $t_3 = 10.767 \cdot 10^{-4}$ sec.

Fig. 3. Functions $P(t)$ for $\varphi_1 = 0.1$ rad: $P_1 = 18.5$ kN, $P_2 = 51.6$ kN, $P_3 = 54.4$ kN, $t_1 = 4.165 \cdot 10^{-5}$ sec, $t_2 = 10.361 \cdot 10^{-4}$ sec, and $t_3 = 10.767 \cdot 10^{-4}$ sec.

For α, we use the following elastoplastic model [4]:

$$\alpha = \begin{cases} bP^{2/3}, & dP/dt > 0,\ P_{\max} < P_1, \\ b_f P^{2/3} + \alpha_p(P_{\max}), & dP/dt < 0,\ P_{\max} > P_1, \\ (1+\beta)c_1 P^{1/2} + (1-\beta)Pd, & dP/dt > 0,\ P_{\max} > P_1. \end{cases} \tag{9}$$

Here $b = (9/(16E^2R))^{1/3}$, $E = E_1E_2((1-\nu_1^2)E_2 + (1-\nu_2^2)E_1)^{-1}$, $R^{-1} = R_2^{-1} - R_1^{-1}$, $P_1 = \chi^3(3R/(4E))^2$, $\chi = \pi k\lambda$, k is the least of the two plastic constants of the colliding bodies, $\lambda = 5.7$, $b_f = R_f^{-1/3}(3/(4E))^{2/3}$, $R_f = (4/3)EP_{\max}^{1/2}\chi^{-3/2}$, $\alpha_p(P_{\max}) = (1-\beta)P_{\max}(2\chi R_p)^{-1}$, $R_p^{-1} = R^{-1} - R_f^{-1}$, $\beta = 0.33$, $c_1 = 3\chi^{1/2}(8E)^{-1}$, and $d = (2\chi R)^{-1}$.

Substituting (2), (8), and (9) into (1), we arrive at the nonlinear integral equation for $P(t)$, which is solved using the Timoshenko iterative procedure [1, 5].

Numerical solutions of the problem are plotted as curves of $P(t)$ in Figs. 1–3 for the following values of the problem parameters: $R_1 = 1$ m, $h = 0.01$ m, $\varphi_0 = 90°$, $R_2 = 0.02$ m, and $m = 0.25$ kg; the material is steel. The velocity is $V_0 = 10$ m/sec. Curve 1 in the figures corresponds to an axisymmetric impact, 2 to a nonaxisymmetric impact, and 3 to an impact on a half-space.

It can be seen from the plots that as the impact angle φ_1 tends to zero, the interaction force $P(t)$ in the nonaxisymmetric case approaches the value of $P(t)$ in the axisymmetric case. Conversely, as φ_1 tends to φ_0, the plots for the cases of a nonaxisymmetric impact and an impact on a half-space approach one another. Physically, this implies that as the impact point approaches the fixing point, the shell behaves itself more rigidly.

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